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Original Research Article

Application of Decomposition of Cartesian Product and Corona Product of Sunlet and Path in Pharmacy

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Abstract: For any positive integer k≥3, we define the sunlet graph of order 2k, denoted by L_2k, as the graph consisting of a cycle of length k together with k pendant vertices such that each pendant vertex is adjacent to exactly one vertex of the cycle so that the degree of each vertex in the cycle is 3. In this paper, we show the necessary and sufficient condition for the decomposition of cartesian product and corona product of sunlet and path into paws, stars, cycles, claws and paths. 2020 Mathematics Subject Classification: 05C38, 05C51, 05C76.

Keywords: Graph theory, decomposition, cartesian product, corona product, sunlet, cycle, path, stars.

INTRODUCTION

All graphs considered here are finite and undirected unless otherwise stated. Let P_r be the path on r vertices. For a graph G, if $E(G)$ can be partitioned into $E_1, E_2, \dots, E_m$ such that the subgraph of G induced by E_i is H_i for all $1 \leq i \leq m$, then $H_1, H_2, \dots, H_m$ decompose G and is written as $E(G) = E(H_1) \cup E(H_2) \cup \dots \cup E(H_m)$. If for $1 \leq i \leq m$, $H_i \cong H$, we say that G has a H-decomposition. In [6], Tay-Woei Shyu studied the decomposition of graphs into paths and stars. In [4], S. Arumugam, I. Sahul Hamid and V. M. Abraham gave the decomposition of a graph G on n vertices (not necessarily connected) into $\left\lfloor \frac{n}{2} \right\rfloor$ paths and cycles. For any two graphs G and H, their cartesian product, denoted by $G \quad H$ has vertex set $V(G \quad H) = V(G) \times V(H)$ and edge set $E(G \quad H) = \{(g,h)(g',h') : g = g',\ hh' \in E(H),\ or\ gg' \in E(G),\ h = h'\}$. It is well known that the cartesian product is commutative and associative. R. Frucht and F. Harary in their paper [3], introduced a new and simple operation on two graphs G_1 and G_2 called their corona product. For any two graphs G_1 and G_2 of orders m and n respectively, their corona product G_1 ⍰ G_2 is the graph obtained by taking one copy of G_1 and m copies of G_2 such that the i^{th} vertex of G_1 is connected to every vertex in the i^{th} copy of G_2. K. Sowndhariya and A. Muthusamy [1], studied the cartesian product of complete graphs into sunlet graphs of order eight. In [5], S. Vimala Roshni and P. Chithra Devi gave the P_4- decomposability of cartesian product of paths and cycles. Here in this paper, we focus on the decomposition of cartesian product and corona product of sunlet and path.

Remark 1.1.

Consider a graph H_1 with $V(H_1) = \{x_{i,j}: 1 \leq i \leq 6;\ 1 \leq j \leq 3\}$ and $E(H_1) = \{x_{i,j}x_{i,j+1}, x_{k,l}x_{k+1,l}, x_{1,q}x_{3,q}, x_{r,s}x_{r+3,s}: 1 \leq i \leq 6; 1 \leq j \leq 2;\ \ 1 \leq k \leq 5;\ \ 1 \leq l, q, r, s \leq 3\}$. H_1 is decomposed into 4 copies of $K_{1,3}$ and 6 copies of $K_{1,2}$ as follows: $\{x_{1,2}x_{1,1}, x_{1,2}x_{2,2}, x_{1,2}x_{3,2}\}, \{x_{1,3}x_{1,2}, x_{1,3}x_{2,3}, x_{1,3}x_{3,3}\}, \{x_{2,2}x_{2,1}, x_{2,2}x_{3,2}, x_{2,2}x_{5,2}\}$, $\{x_{2,3}x_{2,2}, x_{2,3}x_{3,3}, x_{2,3}x_{5,3}\}$, $\{x_{3,2}x_{3,1}, x_{3,2}x_{6,2}\}$, $\{x_{3,3}x_{3,2}, x_{3,3}x_{6,3}\}, \{x_{4,2}x_{4,1}, x_{4,2}x_{1,2}\}$, $\{x_{4,3}x_{4,2}, x_{4,3}x_{1,3}\}, \{x_{5,2}x_{5,1}, x_{5,2}x_{5,3}\}$ and $\{x_{6,2}x_{6,1}, x_{6,2}x_{6,3}\}$.

Theorem 1.2. *For any two positive integers m, n with $m = 3$, $n \geq 2$ and n is even,*
$L_{2m}\square\, P_n$ is decomposed into p copies of $K_{1,3}$ and q copies of $K_{1,2}$ if and only if
$p + \frac{2}{3}q = \frac{5}{3}mn - (n+2)$.

Proof:

Let $G = L_{2m}\square\, P_n$ where $V(G) = \{x_{i,j}: 1 \leq i \leq 2m, 1 \leq j \leq n\}$ and

$$E(G) = \{x_{i,j}x_{i,j+1}, x_{k,l}x_{k+1,l}, x_{1,q}x_{m,q}, x_{r,s}x_{r+m,s}: 1 \leq i \leq 2m, 1 \leq j \leq n-1,$$

$1 \leq k \leq m-1, 1 \leq l, q, s \leq n, 1 \leq r \leq m\}$. Suppose that $L_{2m}\square\, P_n$ is decomposed into p copies of $K_{1,3}$ and q copies of $K_{1,2}$.
Since $|E(G)| = 3p + 2q$, we have
$3p + 2q = 3[n(m-1)] + 2(mn-3)$.
$\Rightarrow 3p + 2q = 5mn - 3n - 6$.
Therefore, $p + \frac{2}{3}q = \frac{5}{3}mn - (n+2)$.
Conversely, suppose that $p + \frac{2}{3}q = \frac{5}{3}mn - (n+2)$.
For $n = 2$, the set of edges $\{x_{1,1}x_{1,2}, x_{1,1}x_{2,1}, x_{1,1}x_{3,1}\}, \{x_{1,2}x_{2,2}, x_{1,2}x_{3,2}, x_{1,2}x_{4,2}\}$, $\{x_{2,1}x_{3,1}, x_{2,1}x_{5,1}, x_{2,1}x_{2,2}\}$, $\{x_{3,2}x_{2,2}, x_{3,2}x_{3,1}, x_{3,2}x_{6,2}\}$, $\{x_{4,1}x_{1,1}, x_{4,1}x_{4,2}\}, \{x_{5,2}x_{5,1}, x_{5,2}x_{2,2}\}$ and $\{x_{6,1}x_{3,1}, x_{6,1}x_{6,2}\}$ forms 4 copies of $K_{1,3}$ and 3 copies of $K_{1,2}$. Therefore $L_6\square\, P_2$ is decomposed into 4 copies of $K_{1,3}$ and 3 copies of $K_{1,2}$. For $n \geq 4$, we prove this theorem by induction on n. For $n = 4$, the induced subgraph $< \{x_{i,3}, x_{i,4}\} >$ together with the edge $x_{i,2}x_{i,3}$ forms $H_1\ \forall\ 1 \leq i \leq 2m$. Thus $E(L_{2m}\square P_4) = E(L_{2m}\square P_2) \cup E(H_1)$. Assume that the theorem is true for $n-2$. The induced subgraph $< \{x_{i,j}, x_{i,j+1}\} >$ together with the edge $x_{i,j-1}x_{i,j}$ where $j = 5, 7, \ldots, n-1$ forms $H_1\ \forall\ 1 \leq i \leq 2m$.

Thus $E(L_{2m}\square P_n) = E(L_{2m}\square P_{n-2}) \cup E(H_1)$. Therefore by induction hypothesis, $L_{2m}\square\, P_{n-2}$ is decomposed into $2(n-2)$ copies of $K_{1,3}$ and $3(n-3)$ copies of $K_{1,2}$. Also by Remark 1.1., H_1 is decomposed into 4 copies of $K_{1,3}$ and 6 copies of $K_{1,2}$. Hence $L_{2m}\square\, P_n$ is decomposed into p copies of $K_{1,3}$ and q copies of $K_{1,2}$.

Remark 1.3.
Consider a graph H_2 with $V(H_2) = \{x_i, y_j: 1 \leq i \leq 4;\ 1 \leq j \leq 2\}$ and $E(H_2) = \{x_ix_{i+1}, x_4x_1, x_jy_j: 1 \leq i \leq 4; 1 \leq j \leq 2\}$. H_2 is decomposed into 3 copies of $K_{1,2}$ as follows: $\{x_1y_1, x_1x_4\}, \{x_3x_2, x_3x_4\}$ and $\{x_2x_1, x_2y_2\}$.

Remark 1.4.
Consider a graph H_3 with $V(H_3) = \{x_{i,j}: 1 \leq i \leq 3;\ 1 \leq j \leq 3\}$ and $E(H_3) = \{x_{i,j}x_{i,j+1}, x_{1,l}x_{2,l}, x_{1,q}x_{3,q}: i = 1, 3;\ 1 \leq j \leq 2;\ 2 \leq l \leq 3;\ 2 \leq q \leq 3\}$.
H_3 is decomposed into 2 copies of $K_{1,3}$ and 1 copy of $K_{1,2}$ as follows: $\{x_{1,2}x_{1,1}, x_{1,2}x_{2,2}, x_{1,2}x_{3,2}\}$, $\{x_{1,3}x_{1,2}, x_{1,3}x_{2,3}, x_{1,3}x_{3,3}\}$ and $\{x_{3,2}x_{3,1}, x_{3,2}x_{3,3}\}$.

Theorem 1.5. *For any two positive integers m, n with $m = 4$, $n \geq 2$ and n is even,*
$L_{2m}\square\, P_n$ is decomposed into p copies of $K_{1,3}$ and q copies of $K_{1,2}$ if and only if
$p + \frac{2}{3}q = \frac{4}{3}(mn-2)$.

Proof:

Let $G = L_{2m}\square\, P_n$ where $V(G) = \{x_{i,j}: 1 \leq i \leq 2m, 1 \leq j \leq n\}$ and

$$E(G) = \{x_{i,j}x_{i,j+1}, x_{k,l}x_{k+1,l}, x_{1,q}x_{m,q}, x_{r,s}x_{r+m,s}: 1 \leq i \leq 2m, 1 \leq j \leq n-1,$$

$1 \leq k \leq m-1, 1 \leq l, q, s \leq n, 1 \leq r \leq m\}$. Suppose that $L_{2m}\square\, P_n$ is decomposed into p copies of $K_{1,3}$ and q copies of $K_{1,2}$.
Since $|E(G)| = 3p + 2q$,
we have $3p + 2q = 3[n(m-1) - 2] + 2[(m+3)\frac{n}{2} - 1]$.
$\Rightarrow 3p + 2q = 4mn - 8$.
$\Rightarrow 3p + 2q = 4(mn-2)$.
Therefore, $p + \frac{2}{3}q = \frac{4}{3}(mn-2)$.

Conversely, suppose that $p + \frac{2}{3}q = \frac{4}{3}(mn-2)$.
For $n = 2$, the induced subgraph $< \{x_{3,1}, x_{3,2}, x_{7,1}, x_{7,2}\} >$ together with the edges $\{x_{3,1}x_{4,1}, x_{2,2}, x_{3,2}\}$ forms H_2. Thus $E(L_8\square P_2) = E(L_6\square P_2) \cup E(H_2)$. Therefore, by Theorem 1.2., $L_6\square P_2$ is decomposed into 4 copies of $K_{1,3}$ and 3 copies of $K_{1,2}$. Also, by Remark 1.3., H_2 is decomposed 3 copies of $K_{1,2}$. For $n \geq 4$, we prove this theorem by induction on n.

For $n = 4$, the induced subgraph $< \{x_{i,3}, x_{i,4}, x_{m-1,3}, x_{m-1,4}, x_{m,3}, x_{m,4}, x_{m+1,3}, x_{m+1,4}, x_{2m-1,3}, x_{2m-1,4}, x_{2m,3}, x_{2m,4}\} >$ together with the edges $\{x_{i,2}x_{i,3}, x_{i+1,3}x_{i,3}, x_{i,4}x_{i+1,4}, x_{m-1,2}x_{m-1,3}, x_{m,2}x_{m,3}, x_{m+1,2}x_{m+1,3}, x_{2m-1,2}x_{2m-1,3}, x_{2m,2}x_{2m,3}\}$ forms H_1
$\forall\ 1 \leq i \leq 2m$. Also, the induced subgraph $< \{x_{i,3}, x_{i,4}, x_{i+m,3}, x_{i+m,4}\} >$ together with the edges $\{x_{i,2}x_{i,3}, x_{i,3}x_{i+1,3}, x_{i,4}x_{i+1,4}, x_{i+m,2}x_{i+m,3}\}$ forms $H_3\ \forall\ i = 2$.
Thus $E(L_{2m}\square P_4) = E(L_{2m}\square P_2) \cup E(H_1) \cup E(H_3)$.

Assume that the theorem is true for $n-2$. The induced subgraph $<\{x_{i,j}, x_{i,j+1}, x_{m-1,j}, x_{m-1,j+1}, x_{m,j}, x_{m,j+1}, x_{m+1,j}, x_{m+1,j+1}, x_{2m-1,j}, x_{2m-1,j+1}, x_{2m,j}, x_{2m,j+1}\}>$ together with the edges $\{x_{i,j-1}x_{i,j}, x_{i+1,j}x_{i,j}, x_{i,j+1}x_{i+1,j+1}, x_{m-1,j-1}x_{m-1,j}, x_{m,j-1}x_{m,j}, x_{m+1,j-1}x_{m+1,j}, x_{2m-1,j-1}x_{2m-1,j}, x_{2m,j-1}x_{2m,j}\}$ where $j = 5, 7, \ldots, n-1$ forms H_1 $\forall\ 1 \le i \le 2m$. Also, the induced subgraph $<\{x_{i,j}, x_{i,j+1}, x_{i+m,j}, x_{i+m,j+1}\}>$ together with the edges $\{x_{i,j-1}x_{i,j}, x_{i,j}x_{i+1,j}, x_{i,j+1}x_{i+1,j+1}, x_{i+m,j-1}x_{i+m,j}\}$ where $j = 5, 7, \ldots, n-1$ forms H_3 $\forall$ $i = 2$.

Thus $E(L_{2m}\square P_n) = E(L_{2m}\square P_{n-2}) \cup E(H_1) \cup E(H_3)$. Therefore, by induction hypothesis, $L_{2m}\square P_{n-2}$ is decomposed into $3n-8$ copies of $K_{1,3}$ and $\frac{7n}{2}$ copies of $K_{1,2}$. Also, by Remark 1.1., H_1 is decomposed into 4 copies of $K_{1,3}$ and 6 copies of $K_{1,2}$. And also, by Remark 1.3., H_3 is decomposed into 2 copies of $K_{1,3}$ and 1 copy of $K_{1,2}$. Hence $L_{2m}\square P_n$ is decomposed into p copies of $K_{1,3}$ and q copies of $K_{1,2}$.

Theorem 1.6. *For any two positive integers m, n with $m = 5$, $n \ge 2$ and n is even, $L_{2m}\square P_n$ is decomposed into p copies of $K_{1,3}$ and q copies of $K_{1,2}$ if and only if* $p + \frac{2}{3}q = \frac{5}{3}[n(m-1)-2]$.

Proof:

Let $G = L_{2m}\square P_n$ where $V(G) = \{x_{i,j}: 1 \le i \le 2m, 1 \le j \le n\}$ and

$E(G) = \{x_{i,j}x_{i,j+1}, x_{k,l}x_{k+1,l}, x_{1,q}x_{m,q}, x_{r,s}x_{r+m,s}: 1 \le i \le 2m, 1 \le j \le n-1,$
$1 \le k \le m-1, 1 \le l, q, s \le n, 1 \le r \le m\}$. Suppose that $L_{2m}\square P_n$ is decomposed into p copies of $K_{1,3}$ and q copies of $K_{1,2}$.

Since $|E(G)| = 3p + 2q$,

we have $3p + 2q = 3[n(m-1)-4] + 2[n(m-1)+1]$.

$\Rightarrow 3p + 2q = 5mn - 5n - 10$.

Therefore, $p + \frac{2}{3}q = \frac{5}{3}[n(m-1)-2]$

Conversely, suppose that $p + \frac{2}{3}q = \frac{5}{3}[n(m-1)-2]$.

For $n = 2$, the induced subgraph $<\{x_{i,j}, x_{i,j+1}, x_{i+m,j}, x_{i+m,j+1}\}>$ together with the edges $\{x_{i-1,j+1}x_{i,j+1}, x_{i,j}, x_{i+1,j}\}$ where $j = 1$ forms 2 copies of H_2 $\forall\ i = 3, 4$.

Thus $E(L_{10}\square P_2) = E(L_6\square P_2) \cup E(H_2) \cup E(H_2)$. Therefore, by Theorem 1.2., $L_6\square P_2$ is decomposed into 4 copies of $K_{1,3}$ and 3 copies of $K_{1,2}$. Also, by Remark 1.3., H_2 is decomposed 6 copies of $K_{1,2}$. For $n \ge 4$, we prove this theorem by induction on n.

For $n = 4$, the induced subgraph $<\{x_{i,3}, x_{i,4}, x_{m-1,3}, x_{m-1,4}, x_{m,3}, x_{m,4}, x_{m+1,3}, x_{m+1,4}, x_{2m-1,3}, x_{2m-1,4}, x_{2m,3}, x_{2m,4}\}>$ together with the edges $\{x_{i,2}x_{i,3}, x_{i+1,3}x_{i,3}, x_{i,4}x_{i+1,4}, x_{m-1,2}x_{m,3}, x_{m+1,2}x_{m+1,3}, x_{2m-1,2}x_{2m-1,3}, x_{2m,2}x_{2m,3}\}$ forms H_1 $\forall\ 1 \le i \le 2m$. Also, the induced subgraph $<\{x_{i,3}, x_{i,4}, x_{i+m,3}, x_{i+m,4}\}>$ together with the edges $\{x_{i,2}x_{i,3}, x_{i,3}x_{i+1,3}, x_{i,4}x_{i+1,4}, x_{i+m,2}x_{i+m,3}\}$ forms 2 copies of H_3 $\forall\ i = 2, 3$.

Thus $E(L_{2m}\square P_4) = E(L_{2m}\square P_2) \cup E(H_1) \cup E(H_3) \cup E(H_3)$. Assume that the theorem is true for $n-2$. The induced subgraph $<\{x_{i,j}, x_{i,j+1}, x_{m-1,j}, x_{m-1,j+1}, x_{m,j}, x_{m,j+1}, x_{m+1,j}, x_{m+1,j+1}, x_{2m-1,j}, x_{2m-1,j+1}, x_{2m,j}, x_{2m,j+1}\}>$ together with the edges $\{x_{i,j-1}x_{i,j}, x_{i+1,j}x_{i,j},$

where $j = 5, 7, \ldots, n-1$ forms H_1 $\forall\ 1 \le i \le 2m$. Also, the induced subgraph $<\{x_{i,j}, x_{i,j+1}, x_{i+m,j}, x_{i+m,j+1}\}>$ together with the edges $\{x_{i,j-1}x_{i,j}, x_{i,j}x_{i+1,j}, x_{i,j+1}x_{i+1,j+1}, x_{i+m,j-1}x_{i+m,j}\}$ where $j = 5, 7, \ldots, n-1$ forms 2 copies of H_3 $\forall\ i = 2, 3$.

Thus $E(L_{2m}\square P_n) = E(L_{2m}\square P_{n-2}) \cup E(H_1) \cup E(H_3) \cup E(H_3)$. Therefore, by induction hypothesis, $L_{2m}\square P_{n-2}$ is decomposed into $4(n-3)$ copies of $K_{1,3}$ and $4n-7$ copies of $K_{1,2}$. Also, by Remark 1.1., H_1 is decomposed into 4 copies of $K_{1,3}$ and 6 copies of $K_{1,2}$. And also, by Remark 1.4., H_3 is decomposed into 4 copies of $K_{1,3}$ and 2 copies of $K_{1,2}$. Hence $L_{2m}\square P_n$ is decomposed into p copies of $K_{1,3}$ and q copies of $K_{1,2}$.

Theorem 1.7. *For any two positive integers m, n with $m \ge 3$, $n \ge 2$ and n is even, $L_{2m}\square P_n$ is decomposed into p copies of $K_{1,3}$ and q copies of $K_{1,2}$ if and only if*

$$p + \frac{2q}{3} = \begin{cases} \frac{n(5m-3)-6(m+2)}{3}, & if\ m = 3 \\ \frac{2[m(2n-3)+8]}{3}, & if\ m \ge 4\ and\ m\ is\ even \\ \frac{5n(m-1)-2(3m-10)}{3}, & if\ m \ge 5\ and\ m\ is\ odd \end{cases}$$

Proof:

Let $G = L_{2m}\square P_n$ where $V(G) = \{x_{i,j}: 1 \le i \le 2m, 1 \le j \le n\}$ and

$E(G) = \{x_{i,j}x_{i,j+1}, x_{k,l}x_{k+1,l}, x_{1,q}x_{m,q}, x_{r,s}x_{r+m,s}: 1 \le i \le 2m, 1 \le j \le n-1,$
$1 \le k \le m-1;\ 1 \le l \le n;\ 1 \le q \le n;\ 1 \le r \le m;\ 1 \le s \le n\}$. Suppose that $L_{2m}\square P_n$ is decomposed into p copies of $K_{1,3}$ and q copies of $K_{1,2}$.

Case (i): $m = 3$

Since $|E(G)| = 3p + 2q$,

we have $3p + 2q = 3[n(m-1) - 2(m-3)] + 2[mn-3]$.

$\Rightarrow 3p + 2q = 5mn - 3n - 6m + 12$.

Therefore, $p + \frac{2q}{3} = \frac{n(5m-3)-6(m+2)}{3}$.

Case (ii): $m \ge 4$ and m is even

Since $|E(G)| = 3p + 2q$,

we have $3p + 2q = 3[n(m-1) - 2(m-3)] +$

$2[\frac{(m+3)n}{2} - 1]$.
$\Rightarrow 3p + 2q = 4mn - 6m + 16$.
Therefore, $p + \frac{2q}{3} = \frac{2[m(2n-3)+8]}{3}$.
Case (iii): $m \geq 5$ and m is odd
Since $|E(G)| = 3p + 2q$,
we have $3p + 2q = 3[n(m-1) - 2(m-3)] + 2[n(m-1) + 1]$.
$\Rightarrow 3p + 2q = 5mn - 5n - 6m + 20$.
Therefore, $p + \frac{2q}{3} = \frac{5n(m-1)-2(3m-10)}{3}$.
Conversely,
Case (i): Suppose that $p + \frac{2q}{3} = \frac{n(5m-3)-6(m+2)}{3}$, if $m = 3$
The proof of this case follows precisely from Theorem 1.2.
Case (ii): Suppose that $p + \frac{2q}{3} = \frac{2[m(2n-3)+8]}{3}$, if $m \geq 4$ and m is even
For $m = 4$, The proof follows precisely from Theorem 1.5. For $m \geq 6$, we prove this case by double induction method.
Subcase (i): $m = 6$
For $n = 2$, the induced subgraph $< \{x_{i,j}, x_{i,j+1}, x_{i+m,j}, x_{i+m,j+1}\} >$ together with the edges $\{x_{i-1,j+1}x_{i,j+1}, x_{i,j}, x_{i+1,j}\}$ where $j = 1$ forms 3 copies of H_2 $\forall\, i = 3, 4, 5$.
Thus $E(L_{2m} \square P_2) = E(L_6 \square P_2) \cup E(H_2) \cup E(H_2) \cup E(H_2)$. Therefore, by Theorem 1.2., $L_6 \square P_2$ is decomposed into 4 copies of $K_{1,3}$ and 3 copies of $K_{1,2}$. Also, by Remark 1.3., H_2 is decomposed 9 copies of $K_{1,2}$. For $n \geq 4$, we prove this theorem by induction on n.
For $n = 4$, the induced subgraph $< \{x_{i,3}, x_{i,4}, x_{m-1,3}, x_{m-1,4}, x_{m,3}, x_{m,4}, x_{m+1,3}, x_{m+1,4}, x_{2m-1,3}, x_{2m-1,4}, x_{2m,3}, x_{2m,4}\} >$ together with the edges $\{x_{i,2}x_{i,3}, x_{i+1,3}x_{i,3}, x_{i,4}x_{i+1,4}, x_{m-1,2}x_{m-1,3}, x_{m,2}x_{m,3}, x_{m+1,2}x_{m+1,3}, x_{2m-1,2}x_{2m-1,3}, x_{2m,2}x_{2m,3}\}$ forms H_1
$\forall\ 1 \leq i \leq 2m$. Also, the induced subgraph $< \{x_{i,3}, x_{i,4}, x_{i+m,3}, x_{i+m,4}\} >$ together with the edges $\{x_{i,2}x_{i,3}, x_{i,3}x_{i+1,3}, x_{i,4}x_{i+1,4}, x_{i+m,2}x_{i+m,3}\}$ forms 3 copies of H_3 $\forall\, i = 2, 3, 4$.
Thus $E(L_{2m} \square P_4) = E(L_{2m} \square P_2) \cup E(H_1) \cup E(H_3) \cup E(H_3) \cup E(H_3)$. Therefore, by Remark 1.1., H_1 is decomposed into 4 copies of $K_{1,3}$ and 6 copies of $K_{1,2}$. Also, by Remark 1.4., H_3 is decomposed into 6 copies of $K_{1,3}$ and 3 copies of $K_{1,2}$. Assume that this subcase is true for $n - 2$. The induced subgraph $< \{x_{i,j}, x_{i,j+1}, x_{m-1,j}, x_{m-1,j+1}, x_{m,j}, x_{m,j+1}, x_{m+1,j}, x_{m+1,j+1}, x_{2m-1,j}, x_{2m-1,j+1}, x_{2m,j}, x_{2m,j+1}\} >$ together with the edges $\{x_{i,j-1}x_{i,j}, x_{i+1,j}x_{i,j}, x_{i,j+1}x_{i+1,j+1}, x_{m-1,j-1}x_{m-1,j}, x_{m,j-1}x_{m,j}, x_{m+1,j-1}x_{m+1,j}, x_{2m-1,j-1}x_{2m-1,j}, x_{2m,j-1}x_{2m,j}\}$ where $j = 5, 7, \ldots, n-1$ forms H_1 $\forall\, 1 \leq i \leq 2m$. Also, the induced subgraph
$< \{x_{i,j}, x_{i,j+1}, x_{i+m,j}, x_{i+m,j+1}\} >$ together with the edges $\{x_{i,j-1}x_{i,j}, x_{i,j}x_{i+1,j}, x_{i,j+1}x_{i+1,j+1}, x_{i+m,j-1}x_{i+m,j}\}$ where $j = 5, 7, \ldots, n-1$ forms 3 copies of H_3 $\forall\, i = 2, 3, 4$.
Thus $E(L_{2m} \square P_n) = E(L_{2m} \square P_{n-2}) \cup E(H_1) \cup E(H_3) \cup E(H_3) \cup E(H_3)$. Therefore, by induction hypothesis, $L_{2m} \square P_{n-2}$ is decomposed into $5n - 16$ copies of $K_{1,3}$ and $\frac{9(n-2)}{2} - 1$ copies of $K_{1,2}$. Hence the Subcase (i) is decomposed into $5n - 1$ copies of $K_{1,3}$ and $\frac{9n}{2} - 1$ copies of $K_{1,2}$. Assume that this Case (ii) is true for $m - 2$. The induced subgraph
$< \{x_{i,j}, x_{i,j+1}, x_{i+m,j}, x_{i+m,j+1}\} >$ together with the edges $\{x_{i,j-1}x_{i,j}, x_{i,j}x_{i+1,j}, x_{i,j+1}x_{i+1,j+1}, x_{i+m,j-1}x_{i+m,j}\}$ where $j = n-3, n-1$ forms 2 copies of H_3
for all $i = m-3, m-2$. Thus $E(L_{2m} \square P_n) = E(L_{2m-2} \square P_n) \cup E(H_3) \cup E(H_3)$.
Therefore, by induction hypothesis, $L_{2m-2} \square P_n$ is decomposed into $n(m-3) - 2(m-5)$ copies of $K_{1,3}$ and $\frac{(m+1)n}{2} - 1$ copies of $K_{1,2}$. Hence $L_{2m} \square P_n$ is decomposed into
$n(m-1) - 2(m-3)$ copies of $K_{1,3}$ and $\frac{(m+3)n}{2} - 1$ copies of $K_{1,2}$ in this case.
Case (iii): Suppose that $p + \frac{2q}{3} = \frac{5n(m-1)-2(3m-10)}{3}$, if $m \geq 5$ and m is odd
For $m = 5$, The proof follows precisely from Theorem 1.6. For $m \geq 7$, we prove this case by double induction method.
Subcase (ii): $m = 7$
For $n = 2$, the induced subgraph $< \{x_{i,j}, x_{i,j+1}, x_{i+m,j}, x_{i+m,j+1}\} >$ together with the edges $\{x_{i-1,j+1}x_{i,j+1}, x_{i,j}, x_{i+1,j}\}$ where $j = 1$ forms 4 copies of H_2 $\forall\, i = 3, 4, 5, 6$.
Thus $E(L_{2m} \square P_2) = E(L_6 \square P_2) \cup E(H_2) \cup E(H_2) \cup E(H_2) \cup E(H_2)$. Therefore, by Theorem 1.2., $L_6 \square P_2$ is decomposed into 4 copies of $K_{1,3}$ and 3 copies of $K_{1,2}$. Also, by Remark 1.3., H_2 is decomposed 12 copies of $K_{1,2}$. For $n \geq 4$, we prove this theorem by induction on n. For $n = 4$, the induced subgraph $< \{x_{i,3}, x_{i,4}, x_{m-1,3}, x_{m-1,4}, x_{m,3}, x_{m,4}, x_{m+1,3}, x_{m+1,4}, x_{2m-1,3}, x_{2m-1,4}, x_{2m,3}, x_{2m,4}\} >$ together with the edges $\{x_{i,2}x_{i,3},$
forms H_1 $\forall\ 1 \leq i \leq 2m$. Also, the induced subgraph $< \{x_{i,3}, x_{i,4}, x_{i+m,3}, x_{i+m,4}\} >$ together with the edges $\{x_{i,2}x_{i,3}, x_{i,3}x_{i+1,3}, x_{i,4}x_{i+1,4}, x_{i+m,2}x_{i+m,3}\}$ forms 4 copies of H_3
for all $i = 2, 3, 4, 5$.
Thus $E(L_{2m} \square P_4) = E(L_{2m} \square P_2) \cup E(H_1) \cup E(H_3) \cup E(H_3) \cup E(H_3) \cup E(H_3)$.
Therefore, by Remark 1.1., H_1 is decomposed into 4 copies of $K_{1,3}$ and 6 copies of $K_{1,2}$. Also, by Remark 1.4., H_3 is decomposed into 8 copies of $K_{1,3}$ and 4 copies of $K_{1,2}$. Assume that this Subcase is true for $n - 2$. The induced subgraph $< \{x_{i,j}, x_{i,j+1}, x_{m-1,j}, x_{m-1,j+1}, x_{m,j}, x_{m,j+1}, x_{m+1,j}, x_{m+1,j+1}, x_{2m-1,j}, x_{2m-1,j+1}, x_{2m,j}, x_{2m,j+1}\} >$ together with the edges $\{x_{i,j-1}x_{i,j},$

$x_{i+1,j}x_{i,j}, x_{i,j+1}x_{i+1,j+1}, x_{m-1,j-1}x_{m-1,j}, x_{m,j-1}x_{m,j}, x_{m+1,j-1}x_{m+1,j},$ $x_{2m,j-1}x_{2m,j}\}$ where $j = 5, 7, \ldots, n-1$ forms H_1 $\forall\, 1 \le i \le 2m$. Also, the induced subgraph $< \{x_{i,j}, x_{i,j+1}, x_{i+m,j}, x_{i+m,j+1}\} >$ together with the edges $\{x_{i,j-1}x_{i,j}, x_{i,j}x_{i+1,j}, x_{i,j+1}x_{i+1,j+1},$ $x_{i+m,j-1}x_{i+m,j}\}$ where $j = 5, 7, \ldots, n-1$ forms 4 copies of H_3 $\forall\, i = 2, 3, 4, 5$.

Thus $E(L_{2m}\square P_n) = E(L_{2m}\square P_{n-2}) \cup E(H_1) \cup E(H_3) \cup E(H_3) \cup E(H_3) \cup E(H_3)$.

Therefore, by induction hypothesis, $L_{2m}\square P_{n-2}$ is decomposed into $2(3n-10)$ copies of $K_{1,3}$ and $6n-11$ copies of $K_{1,2}$. Hence the Subcase (ii) is decomposed into $2(3n-4)$ copies of $K_{1,3}$ and $6n+1$ copies of $K_{1,2}$. Assume that this Case (iii) is true for $m-2$.

The induced subgraph $< \{x_{i,j}, x_{i,j+1}, x_{i+m,j}, x_{i+m,j+1}\} >$ together with the edges $\{x_{i,j-1}x_{i,j},$ $x_{i,j}x_{i+1,j}, x_{i,j+1}x_{i+1,j+1}, x_{i+m,j-1}x_{i+m,j}\}$ where $j = n-3, n-1$ forms 2 copies of H_3 for all $i = m-3, m-2$. Thus $E(L_{2m}\square P_n) = E(L_{2m-2}\square P_n) \cup E(H_3) \cup E(H_3)$.

Therefore, by induction hypothesis, $L_{2m-2}\square P_n$ is decomposed into $n(m-3) - 2(m-5)$ copies of $K_{1,3}$ and $n(m-3)+1$ copies of $K_{1,2}$. Hence $L_{2m}\square P_n$ is decomposed into $n(m-1) - 2(m-3)$ copies of $K_{1,3}$ and $n(m-1)+1$ copies of $K_{1,2}$ in this case.

Therefore, $L_{2m}\square P_n$ is decomposed into p copies of $K_{1,3}$ and q copies of $K_{1,2}$.

Theorem 1.8. *For any two positive integers m, n with $m = 3$, $n \ge 3$ and n is odd, $L_{2m}\square P_n$ is decomposed into p copies of $K_{1,3}$, q copies of $K_{1,2}$ and one copy of P_4 if and only if $p + \frac{2}{3}q = \frac{m}{3}(5n-4) - (n-1)$.*

Proof:

Let $G = L_{2m}\square P_n$ where $V(G) = \{x_{i,j} : 1 \le i \le 2m;\ 1 \le j \le n\}$ and

$$E(G) = \{x_{i,j}x_{i,j+1}, x_{k,l}x_{k+1,l}, x_{1,q}x_{m,q}, x_{r,s}x_{r+m,s} : 1 \le i \le 2m; 1 \le j \le n-1;$$
$$1 \le k \le m-1, 1 \le l \le n;\ 1 \le q \le n, 1 \le r \le m, 1 \le s \le n\}.$$

Suppose that $L_{2m}\square P_n$ is decomposed into p copies of $K_{1,3}$, q copies of $K_{1,2}$ and one copy of P_4.

Since $|E(G)| = 3p + 2q + 3$,

we have $3p + 2q + 3 = 3[n(m-1)+1] + 2(mn-2m) + 3$.

$\Rightarrow 3p + 2q + 3 = 5mn - 4m - 3n + 6$.

$\Rightarrow 3(p+n-1) + 2q = m(5n-4)$.

Therefore, $p + \frac{2}{3}q = \frac{m}{3}(5n-4) - (n-1)$.

Conversely, suppose that $p + \frac{2}{3}q = \frac{m}{3}(5n-4) - (n-1)$. We prove this theorem by induction on n. For $n = 3$, the set of edges $\{x_{1,1}x_{1,2}, x_{1,1}x_{2,1}, x_{1,1}x_{4,1}\}$, $\{x_{1,3}x_{2,3}, x_{1,3}x_{3,3}, x_{1,3}x_{4,3}\}, \{x_{3,1}x_{1,1}, x_{3,1}x_{3,2}, x_{3,1}x_{6,1}\}$, $\{x_{3,2}x_{1,2}, x_{3,2}x_{3,3}, x_{3,2}x_{6,2}\}$, $\{x_{5,2}x_{5,1}, x_{5,2}x_{5,3}\}, \{x_{4,2}x_{4,1}, x_{4,2}x_{4,3}\}$, $\{x_{2,3}x_{3,3}, x_{2,3}x_{5,3}\}$ and $\{x_{3,3}x_{6,3}, x_{6,3}x_{6,2}, x_{6,2}x_{6,1}\}$ forms 7 copies of $K_{1,3}$, 3 copies of $K_{1,2}$ and one copy of P_4. Therefore $L_6\square P_3$ is decomposed into 7 copies of $K_{1,3}$, 3 copies of $K_{1,2}$ and one copy of P_4. For $n \ge 5$, we prove this theorem by induction on n. For $n = 5$, the induced subgraph $< \{x_{i,4}, x_{i,5}\} >$ together with the edge $x_{i,3}x_{i,4}$ forms H_1 $\forall\, 1 \le i \le 2m$. Thus $E(L_{2m}\square P_5) = E(L_{2m}\square P_3) \cup E(H_1)$. Assume that the theorem is true for $n-2$. The induced subgraph $< \{x_{i,j}, x_{i,j+1}\} >$ together with the edge $x_{i,j-1}x_{i,j}$ where $j = 4, 6, \ldots, n-1$ forms H_1 $\forall\, 1 \le i \le 2m$.

Thus $E(L_{2m}\square P_n) = E(L_{2m}\square P_{n-2}) \cup E(H_1)$.

Therefore, by induction hypothesis, $L_{2m}\square P_{n-2}$ is decomposed into $2n-3$ copies of $K_{1,3}$, $3(n-4)$ copies of $K_{1,2}$ and one copy of P_4. Also, by Remark 1.1., H_1 is decomposed into 4 copies of $K_{1,3}$ and 6 copies of $K_{1,2}$. Hence $L_{2m}\square P_n$ is decomposed into p copies of $K_{1,3}$, q copies of $K_{1,2}$ and one copy of P_4.

Remark 1.9.

Consider a graph H_4 with $V(H_4) = \{x_{i,j} : 1 \le i \le 3;\ 1 \le j \le 3\}$ and

$$E(H_4) = \{x_{i,j}x_{i,j+1}, x_{1,l}x_{2,l}, x_{1,q}x_{3,q} : i = 1,3; 1 \le j \le 2; 1 \le l \le 3;\ 1 \le q \le 3\}.$$

H_4 is decomposed into 2 copies of $K_{1,3}$ and 2 copies of $K_{1,2}$ as follows:

$\{x_{1,1}x_{1,2}, x_{1,1}x_{2,1}, x_{1,1}x_{3,1}\}$, $\{x_{1,2}x_{2,2}, x_{1,2}x_{1,3}, x_{1,2}x_{3,2}\}$, $\{x_{1,3}x_{2,3}, x_{1,3}x_{3,3}\}$ and $\{x_{3,2}x_{3,1}, x_{3,2}x_{3,3}\}$.

Theorem 1.10. *For any two positive integers m, n with $m \ge 3$, $n \ge 3$ and n is odd, $L_{2m}\square P_n$ is decomposed into p copies of $K_{1,3}$, q copies of $K_{1,2}$ and one copy of P_4 if and only if $p + \frac{2}{3}q = \frac{2m(2n-1)}{3} - 1$.*

Proof:

Let $G = L_{2m}\square P_n$ where $V(G) = \{x_{i,j} : 1 \le i \le 2m;\ 1 \le j \le n\}$ and

$$E(G) = \{(x_{i,j}x_{i,j+1}, x_{k,l}x_{k+1,l}, x_{1,q}x_{m,q}, x_{r,s}x_{r+m,s}) : 1 \le i \le 2m; 1 \le j \le n-1;$$
$$1 \le k \le m-1;\ 1 \le l \le n;\ 1 \le q \le n;\ 1 \le r \le m;\ 1 \le s \le n\}.$$

Suppose that $L_{2m}\square P_n$ is decomposed into p copies of $K_{1,3}$, q copies of $K_{1,2}$ and one copy of P_4.

Since $|E(G)| = 3p + 2q + 3$,

we have $3p + 2q + 3 = 3[n(m-1) + (4-m)] + 2[\frac{(m+3)(n-1)}{2} + (m-6)] + 3$.

$\Rightarrow 3p + 2q + 3 = 4mn - 2m - 3$.

Therefore, $p + \frac{2}{3}q = \frac{2m(2n-1)}{3} - 1$.

Conversely, suppose that $p + \frac{2}{3}q = \frac{2m(2n-1)}{3} - 1$.

For $n = 3$, the induced subgraph $< \{x_{i,j-1}, x_{i,j}, x_{i,j+1}, x_{i+m-1,j-1}, x_{i+m-1,j}, x_{i+m-1,j+1}\} >$ together with the edges $\{x_{i,j-1}x_{i+1,j-1}, x_{i,j}, x_{i+1,j}, x_{i,j+1}, x_{i+1,j+1}\}$ where $j = 2$ forms $m-3$ copies of H_4 $\forall\, i = 3, 4, \ldots, m-1$.

Thus, $E(L_{2m}\square P_3) = E(L_6\square P_3) \cup [E(H_4) \cup \ldots \cup$

$E(H_4)\ (m-3)\text{ - }times]$. Therefore, by Theorem 1.8., $L_6\square P_3$ is decomposed into 4 copies of $K_{1,3}$ and 3 copies of $K_{1,2}$. Also, by Remark 1.9., H_4 is decomposed $2(m-3)$ copies of $K_{1,2}$ and $2(m-3)$ copies of $K_{1,2}$.

For $n \geq 5$, we prove this theorem by induction on n.

For $n = 5$, the induced subgraph $<\{x_{i,4}, x_{i,5}, x_{m-1,4}, x_{m-1,5}, x_{m,4}, x_{m,5}, x_{m+1,4}, x_{m+1,5}, x_{2m-1,4}, x_{2m-1,5}, x_{2m,4}, x_{2m,5}\}>$ together with the edges $\{x_{i,3}x_{i,4}, x_{i+1,4}x_{i,4}, x_{i,5}x_{i+1,5}, x_{m-1,3}x_{m-1,4}, x_{m,3}x_{m,4}, x_{m+1,3}x_{m+1,4}, x_{2m-1,3}x_{2m-1,4}, x_{2m,3}x_{2m,4}\}$ forms $H_1\ \forall\ 1 \leq i \leq 2m$.

Also, the induced subgraph $<\{x_{i,4}, x_{i,5}, x_{i+m,4}, x_{i+m,5}\}>$ together with the edges $\{x_{i,3}x_{i,4}, x_{i,4}x_{i+1,4}x_{i,5}x_{i+1,5}, x_{i+m,3}x_{i+m,4}\}$ forms $(m-3)$ copies of $H_3\ \forall\ i = 2, 3, \ldots, m-2$.

Thus $E(L_{2m}\square P_5) = E(L_{2m}\square P_3) \cup E(H_1) \cup [E(H_3) \cup \ldots \cup E(H_3)(m-3)\text{ - }times]$

Assume that the theorem is true for $n-2$. The induced subgraph $<\{x_{i,j}, x_{i,j+1}, x_{m-1,j}, x_{m-1,j+1}, x_{m,j}, x_{m,j+1}, x_{m+1,j}, x_{m+1,j+1}, x_{2m-1,j}, x_{2m-1,j+1}, x_{2m,j}, x_{2m,j+1}\}>$ together with the edges $\{x_{i,j-1}x_{i,j}, x_{i+1,j}x_{i,j}, x_{i,j+1}x_{i+1,j+1}, x_{m-1,j-1}x_{m-1,j}, x_{m,j-1}x_{m,j}, x_{m+1,j-1}x_{m+1,j}, x_{2m-1,j-1}x_{2m-1,j}, x_{2m,j-1}x_{2m,j}\}$ where $j = 6, 8, \ldots, n-1$ forms $H_1\ \forall\ 1 \leq i \leq 2m$.

Also, the induced subgraph $<\{x_{i,j}, x_{i,j+1}, x_{i+m,j}, x_{i+m,j+1}\}>$ together with the edges $\{x_{i,j-1}x_{i,j}, x_{i,j}x_{i+1,j}, x_{i,j+1}x_{i+1,j+1}, x_{i+m,j-1}x_{i+m,j}\}$ where $j = 6, 8, \ldots, n-1$ forms $m-3$ copies of $H_3\ \forall\ i = 2, 3, \ldots, m-2$.

Thus $E(L_{2m}\square P_n) = E(L_{2m}\square P_{n-2}) \cup E(H_1) \cup [E(H_3) \cup \ldots \cup E(H_3)\ (m-3)\text{ - }times]$.

Therefore, by induction hypothesis, $L_{2m}\square P_{n-2}$ is decomposed into $(n-2)(m-1) + (4-m)$ copies of $K_{1,3}$, $\frac{(m+3)(n-3)}{2} + (m-6)$ copies of $K_{1,2}$ and one copy of P_4.

Also, by Remark 1.1., H_1 is decomposed into 4 copies of $K_{1,3}$ and 6 copies of $K_{1,2}$.

And also, by Remark 1.4., H_3 is decomposed into $2(m-3)$ copies of $K_{1,3}$ and $(m-3)$ copies of $K_{1,2}$. Hence $L_{2m}\square P_n$ is decomposed into p copies of $K_{1,3}$, q copies of $K_{1,2}$ and one copy of P_4.

Theorem 1.11. *For any two positive integers m, n with $m \geq 3$, $n \geq 2$ and n is even, $L_{2m}\square P_n$ is decomposed into p copies of $K_{1,3} + e$ if and only if $p = mn$.*

Proof:

Let $G = L_{2m}\square P_n$. Let $V(L_{2m}) = \{u_i, v_i : 1 \leq i \leq m\}$, where u_i is the pendant vertex adjacent to v_i. Let $V(G) = \{u_i, v_i, u_{ij'}, v_{ij'} : 1 \leq i \leq m;\ 1 \leq j \leq n\}$, where $u_{i1'}, u_{i2'}, \ldots, u_{in'}$ are the vertices in the copy of the path corresponding to each u_i and $v_{i1'}, v_{i2'}, \ldots, v_{in'}$ are the vertices in the copy of the path corresponding to each v_i. Suppose that G is decomposed into p copies of $K_{1,3} + e$.

Since $|E(G)| = 4p$,

we have $4p = 4mn$.

Therefore, $p = mn$.

Conversely, suppose that $p = mn$.

Now, the induced subgraph $<\{u_{i1'}, u_{i2'}, u_i, v_i\}> \cong K_{1,3} + e\ \forall\ 1 \leq i \leq m$. The induced subgraph $<\{u_{i(j-1)'}, u_{ij'}, u_i\}>$ together with the edge $u_{i(j-2)'}\ u_{i(j-1)'}$, where $j = 4, 6, \ldots, n$ forms $K_{1,3} + e\ \forall\ 1 \leq i \leq m$. Also, the induced subgraph $<\{v_{(i-1)1'}, v_{(i-1)2'}, v_{i-1}, v_i\}> \cong K_{1,3} + e\ \forall\ 2 \leq i \leq m$. The induced subgraph $<\{v_{i(j-1)'}, v_{ij'}, v_i\}>$ together with the edge $v_{i(j-2)'}\ v_{i(j-1)'}$, where $j = 4, 6, \ldots, n$ forms $K_{1,3} + e\ \forall\ 1 \leq i \leq m$. And also, the induced subgraph $<\{v_{m1'}, v_{m2'}, v_m, v_1\}> \cong K_{1,3} + e$. The induced subgraph $<\{v_{mj'}, v_{m(j-1)'}, v_m\}>$ together with the edge $v_{m(j-2)'}\ v_{m(j-1)'}$, where $j = 4, 6, \ldots, n$ forms $K_{1,3} + e$.

Thus $E(G) = [E(K_{1,3} + e) \cup E(K_{1,3} + e) \cup \ldots \cup E(K_{1,3} + e)\ mn\text{ - }times]$. Hence $L_{2m}\square P_n$ is decomposed into p copies of $K_{1,3} + e$.

Corollary 1.12. *For any two positive integers m, n with $m \geq 3$, $n \geq 2$ and n is even, $L_{2m}\square P_n$ is decomposed into p copies of $K_{1,3} + e$ and one copy of S_m if and only if $p = mn + \frac{n}{2}$.*

Theorem 1.13. *For any two positive integers m, n with $m \geq 3$, $n \geq 3$ and n is odd, $L_{2m}\square P_n$ is decomposed into p copies of $K_{1,3} + e$ and q copies of $K_{1,2}$ if and only if $2p + q = 2m$.*

Proof:

Let $G = L_{2m}\square P_n$. Let $V(L_{2m}) = \{u_i, v_i : 1 \leq i \leq m\}$, where u_i is the pendant vertex adjacent to v_i. Let $V(G) = \{u_i, v_i, u_{ij'}, v_{ij'} : 1 \leq i \leq m, 1 \leq j \leq n\}$, where $u_{i1'}, u_{i2'}, \ldots, u_{in'}$ are the vertices in the copy of the path corresponding to each u_i and $v_{i1'}, v_{i2'}, \ldots, v_{in'}$ are the vertices in the copy of the path corresponding to each v_i. Suppose that G is decomposed into p copies of $K_{1,3} + e$ and q copies of $K_{1,2}$.

Since $|E(G)| = 4p + 2q$,

we have $4p + 2q = 4m(n-1) + 2(2m)$.

$\Rightarrow 4p + 2q = 4mn$.

Therefore, $2p + q = mn$.

Conversely, suppose that $2p + q = mn$.

Now, the induced subgraph $<\{u_{i1'}, u_{i2'}, u_i, v_i\}> \cong K_{1,3} + e\ \forall\ 1 \leq i \leq m$. The induced subgraph $<\{u_{i(j-1)'}, u_{ij'}, u_i\}>$ together with the edge $u_{i(j-2)'}\ u_{i(j-1)'}$, where $j = 4, 6, \ldots, n-1$ forms $K_{1,3} + e\ \forall\ 1 \leq i \leq m$. The induced subgraph $<\{u_{i(n-1)'}u_{in'},\ u_{in'}u_i\}> \cong K_{1,2}\ \forall\ 1 \leq i \leq m$. Also, the induced subgraph $<\{v_{(i-1)1'}, v_{(i-1)2'}, v_{i-1}, v_i\}> \cong K_{1,3} + e\ \forall\ 2 \leq i \leq m$. The induced subgraph $<\{v_{ij'}, v_{i(j-1)'}, v_i\}>$ together with the edge $v_{i(j-2)'}\ v_{i(j-1)'}$, where $j = 4, 6, \ldots, n-1$ forms $K_{1,3} + e\ \forall\ 1 \leq i \leq m$. The induced subgraph $<\{v_{i(n-1)'}v_{in'},\ v_{in'}v_i\}> \cong K_{1,2}\ \forall\ 1 \leq i \leq m$. And also, the induced subgraph

$< \{v_{m1'}, v_{m2'}, v_m, v_1\} > \cong K_{1,3} + e$. The induced subgraph $< \{v_{mj'}, v_{m(j-1)'}, v_m\} >$ together with the edge $v_{m(j-2)'}\, v_{m(j-1)'}$, where $j = 4, 6, \dots, n-1$ forms $K_{1,3} + e$. The induced subgraph $< \{v_{m(n-1)'} v_{mn'}, v_{mn'} v_m\} > \cong K_{1,2}$.

Thus $E(G) = [E(K_{1,3} + e) \cup E(K_{1,3} + e) \cup \dots \cup E(K_{1,3} + e)(mn - m)\text{-}times]$

$$\cup\, [E(K_{1,2}) \cup E(K_{1,2}) \cup \dots \cup E(K_{1,2})\ 2m\text{-}times].$$

Hence $L_{2m} \square P_n$ is decomposed into p copies of $K_{1,3} + e$ and q copies of $K_{1,2}$.

Corollary 1.14. *For any two positive integers* m, n *with* $m \geq 3, n \geq 3$ *and* n *is odd,*
$H_m \square P_n$ *is decomposed into* p *copies of* $K_{1,3} + e$, q *copies of* $K_{1,2}$ *and one copy of* S_m
if and only if $p + \frac{q}{2} = \frac{n(m+1)}{2}$.

Applications:

The Cartesian product of graphs provides mathematical framework for complex pharmaceutical systems. It helps in predictive modeling and simulation, supports clinical decision making and also it enhances drug safety and treatment efficiency. The corona product of graphs provides a mathematical support for complex therapeutic modeling. It represents hierarchical and modular pharmaceutical systems, supports advanced drug design and clinical research and also it helps in modeling drug targeting and interaction mechanisms.

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